Group symmetries in two-body random matrix ensembles generating order out of complexity

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The two-body random matrix ensembles with spin TBRE-*s* and in a single *j* shell TBRE-*j* introduced recently in the context of ground state structures in complex interacting particle systems, possess $U(N)\supset U(N/2)\otimes SU(2)$ and $U(N)\supset O(3)$ group symmetries, respectively, with N the number of single particle states. It is shown that both these group symmetries give rise to simplicities in the ground state structures but in different ways.

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I. INTRODUCTION

There is now new interest in investigating two-body random matrix ensembles (TBRE) with various deformations [1]. Early formulation of TBRE is due to French, Bohigas, Flores, and their co-workers [2]. Recently, many research groups have pointed out that TBRE with extra information provide a framework for understanding the structure of levels in the ground state domain of complex isolated finite interacting particle systems such as atomic nuclei, quantum dots, etc. For example, in nuclei the focus is in using the nuclear shell model as a laboratory for deriving or testing various predictions of extended TBRE's for order out of chaos in the ground state domain, localization measures such as the information entropy, Breit-Wigner to Gaussian transition in strength functions, nature of Gamow-Teller and other transition strength sums, and so on [1,3–6]. Similarly, deformed TBRE are used recently in the description of observed conductance peak spacing and peak height distributions in Coulomb blockade quantum dots [7], in the study of the role of interaction fluctuations on their ground state spin [8,9] etc. In the simplest form, TBRE is defined for spinless fermions (or Bosons) and here the interactions, that are two body are represented by a random matrix in two-particle spaces and propagated to many-particle spaces by using the geometry of the space (note that one considers m particles in N single particle states and constructs the m particle states as direct products of single particle states). One class of extension of TBRE is to impose group symmetries on the two-body interaction, they are called TBRE-sym [1]. Two different types of TBRE-sym are discussed in this paper.

Recently, Jacquod and Stone [8,9] and Kaplan *et al.* [10] considered TBRE with spin degree of freedom (called TBRE-*s*) and discussed the ground state magnetization of disordered systems such as quantum dots and the ground state spin structure of a general complex interacting particle system, respectively. On the other hand, Mulhall *et al.* [4] introduced a single *j*-shell model with *m*-particle states having good angular momentum (J) and the two-body random interaction (TBRE-*j*). Using TBRE-*j* a basis is given for the observed dominance of $J=0$ ground states in many nuclear shell model and interacting boson model calculations. The results in [4,8–10] are based on the near Gaussian form of the density of states ($\rho(E)$) in the two models and by deriving easy to understand forms for the centroids, variances, and lower order shape parameters. Towards this end, for the TBRE-*s*, in [8–10] some counting arguments are employed while for the TBRE-*j*, cranking model and Fermi occupancies are used [4]. The purpose of the present paper is to reexamine the results of the two TBRE's from the standpoint of their group structure. First, there is a $U(N)$ group operating in the m-particle spaces. For TBRE with spinless fermions, the relevance of this $U(N)$ group structure is pointed out in a recent publication [5]. With the $U(N)$ group, it is seen that TBRE-*s* and TBRE-*j* possess $U(N)\supset U(N/2)\otimes SU(2)$ and $U(N)\supset O(3)$ group structures, respectively. Due to these different group symmetries, the origin of simplicities in the two cases are quite different. Results for TBRE-*s* are given in Sec. II and for TBRE-*j* in Sec. III. Finally, Sec. IV gives some concluding remarks and future outlook.

II. TWO-BODY RANDOM MATRIX ENSEMBLES WITH SPIN

A. Variances for TBRE-*s* and the ground state structure

Let us consider m fermions in Ω number of single particle (SP) levels each doubly degenerate (thus $N=2\Omega$) with spin $s=\frac{1}{2}$ and $s_z=\pm\frac{1}{2}$. The number of levels $d(m,S)$ with fixed total m-particle spin S is easily determined by considering m_1 particles with $s_z=\frac{1}{2}$ and m_2 with $s_z=-\frac{1}{2}$. The dimension (number of states) for a fixed (m_1,m_2) is simply

$$D(m_1,m_2)=\binom{\Omega}{m_1}\binom{\Omega}{m_2}\Rightarrow D(m,S_z);$$

$$m=m_1+m_2 \quad \text{and} \quad S_z=\frac{(m_1-m_2)}{2}.$$

Now the fixed-S dimension $d(m,S)=D(m,S_z=S)-D(m,S_z=S+1)$ is

$$d(m,S)=\frac{(2S+1)}{(\Omega+1)}\binom{\Omega+1}{m/2+S+1}\binom{\Omega+1}{m/2-S}. \qquad (1)$$

Note that

$$\sum_S (2S+1)d(m,S)=\binom{N}{m}.$$

With S a good quantum number, the Hamiltonian H (which is two body or $(1+2)$ body) has $\mathrm{U}(N)\supset\mathrm{U}(N/2)\otimes\mathrm{SU}(2)$ group symmetry and this is well known [11–13]. All the m-particle states belong to the totally antisymmetric representation of $\mathrm{U}(N)$ and the spin S is generated by the SU(2) group. The direct product group structure immediately gives the result that fixed S averages of any operator will be a polynomial in m and $S(S+1)$; note that m is the eigenvalue of the number operator $\hat{n}$ and $S(S+1)$ is the eigenvalue of the $\hat{S}^2$ operator. This then leads to simple forms for the centroids and variances of fixed (m,S) densities $\rho^{(m,S)}(E)=\langle\delta(H-E)\rangle^{m,S}$. Let us point out that a general two-body Hamiltonian $H=V(2)$ is defined by the two-body matrix elements $V^s_{ijkl}={}_a\langle(ij)s|V(2)|(kl)s\rangle_a$, $s=0,1$ (note that $|(ij)s\rangle_a$ denotes antisymmetrized two-particle state). For a TBRE-s one assumes $\overline{V^s_{ijkl}}=0$ and $\overline{(V^s_{ijkl})^2}=U_s^2$, i.e., the V^s_{ijkl} are zero centered random variables with variance U_s^2 (usually the random variables are taken to be Gaussian in nature). Note that the bar over V's denotes ensemble average. Quite similar to the $\mathrm{U}(N)$ case (Appendix), with respect to the $\mathrm{U}(N/2)\otimes\mathrm{SU}(2)$ group, the $V(2)$ will have scalar ($\nu=0$), effective one-body ($\nu=1$), and irreducible two-body ($\nu=2$) parts in each spin ($s=0,1$) sector [12,14]. The $\nu=0$ parts generate the centroids $\varepsilon(m,S)$ and obviously they will be zero on ensemble average. In the dilute limit defined by $\Omega\to\infty$, $m\to\infty$, and $m/\Omega\to 0$, the $V^{\nu=2,s=0,1}$ parts generate the TBRE-s variances $\sigma^2(m,S)$ [contributions from the $\nu=1$ parts will be smaller at least by the factor (m/Ω)] and then,

$$\sigma^2(m,S)\xrightarrow{\text{TBRE}-s}\frac{[(\Omega-m/2)(\Omega-m/2+1)-S(S+1)][m(m+2)-4S(S+1)]}{8\Omega(\Omega-1)}\{[\Omega(\Omega+1)/2]U_0^2\}$$
$$+\frac{\{S^2(S+1)^2(3\Omega^2-7\Omega+6)/2+3m(m-2)(\Omega-m/2)(\Omega-m/2-1)(\Omega+1)(\Omega+2)/8+[S(S+1)/2][(5\Omega-3)(\Omega+2)(m/2-\Omega)m+\Omega(\Omega-1)(\Omega+1)(\Omega+6)]\}}{\Omega(\Omega+1)(\Omega-2)(\Omega-3)}$$
$$\times[(\Omega(\Omega-1)/2)U_1^2]=P_0(m,S)U_0^2+P_1(m,S)U_1^2. \tag{2}$$

Using Eq. (2) it is seen that the TBRE-s variances get smaller as the spin S is increasing and this trend is independent of the ratio U_1^2/U_0^2. For example, for $\Omega=20$, $m=8$, and $S=(0,1,2,3,4)$ the $\sigma^2(m,S)/\sigma^2(m,0)$ values for $U_1^2/U_0^2=0.3$, 1, and 3 are (1.0, 0.92, 0.77, 0.55, 0.29), (1.0, 0.94, 0.84, 0.69, 0.52), and (1.0, 0.96, 0.88, 0.77, 0.67), respectively. It is worth pointing out that the behavior of the dimensions $d(m,S)$ with respect to S is some what different. They grow from $S=0$ to $S=1$ and then start decreasing fast. For example, for ($\Omega=30,m=10$) the dimensions for $S=0$, 1, 2, 3, 4, and 5 are $\sim 4\times 10^9$, $\sim 8\times 10^9$, $\sim 6\times 10^9$, $\sim 2\times 10^9$, $\sim 4\times 10^8$, and $\sim 3\times 10^7$, respectively. At this stage it is important to discuss the relationship of $P_s(m,S)$ with the so-called connectivity factor $K_s(m,S)$ studied in [9]. Say a typical $|S\alpha\rangle$ state is connected to $K_s(m,S)$ number of states $|S\beta\rangle$ by the $s=0,1$ parts of the two-body interaction and the variances of the connected matrix elements is U_s^2 (independent of β). Then $\sigma^2(m,S)=\Sigma_s K_s(m,S)U_s^2$ and this expression is used by Jacquod and Stone by deriving formulas for $K_s(m,S)$ [9]. Comparing with Eq. (2), it is expected that $P_s(m,S)=K_s(m,S)$. It is seen that Eq. (2) gives the expressions for the factors $P_{s=0,1}(m,S=0)$ and $P_{s=1}(m,S=m/2)$ that are identical to the results for $K_{s=0,1}(m,S=0)$ and $K_{s=1}(m,S=m/2)$ given in Eqs. (B1) and (B2), respectively, of [9]. However, by comparing Eq. (2) with Eq. (B4) of [9] it is clearly seen that for $S\neq(0,m/2)$, $K(m,S)\neq P(m,S)$ (here one is assuming $U_0^2=U_1^2=U^2$). Thus in these situations, the variances are $P(m,S)U^2$ but not $K(m,S)U^2$. One reason for the differences is that the variance of each allowed (nonzero) m-particle matrix element is not U^2 in general but it is a complicated function of the spin S that is generated by m spin $\frac{1}{2}$ couplings.

It is well verified in a number of numerical examples [8,10,11] that $\rho^{m,S}(E)$ is in general a Gaussian with exponential tails [15]. Then the ground state energy (E_g) for a given S follows from the so-called Ratcliff prescription [16] $\frac{1}{2}=\int_{-\infty}^{E_g} d(m,S)\rho^{m,S}(E)dE$. Then $E_g\approx-\sigma(m,S)\ln d(m,S)$ away from the centroid $\varepsilon(m,S)$. The behavior of the fixed S variances (decrease with increasing S) and the logarithmic dependence of E_g on the dimensions, clearly show that in general one gets nearly degenerate $S=0$ and $S=1$ ground states and other S ground states will be far above. Thus the structure of the levels near the ground state for TBRE-s will be dominated by $S=0$ and $S=1$ as already pointed out in [8–10]. The precise structure (whether $S=0$ is the ground state or the $S=1$) depends on the form of the tails of $\rho^{m,S}(E)$. This problem is treated in two different ways in literature. Here we consider Jacquod and Stone approach that is based on the variances and in Sec. II B we will turn to Kaplan *et al.* approach that is based on the excess correction. Jacquod and Stone [8,9] carried out detailed numerical TBRE-s calculations with $H=V(2)$ and observed that $S=0$ ($S=\frac{1}{2}$) is always the ground state for even-m (odd-m) systems. This and the fact that the $S=0$ variances are always larger than $S=1$ variances is used in conjecturing that

$$\Delta\equiv E_{\mathrm{gs}}(S_{\min})-E_{\mathrm{gs}}(S_{\min}+1)=\beta[\sigma(m,S_{\min})-\sigma(m,S_{\min}+1)]$$

where $S_{\min}=0$ or $\frac{1}{2}$, E_{gs} stands for the ground state energy and β is a free parameter. Now employing the relation $\sigma^2(m,S)=K(m,S)U^2$ it is shown that the conjecture describes very well TBRE-*s* numerical results. One of the striking observation is the odd-even effect (with respect to m) in Δ and the $K(m,S)$ formulas are able to reproduce this effect. However, as pointed out before, $\sigma^2(m,S)$ should be in terms of $P(m,S)$ and, therefore, it is more appropriate to use

$$\Delta \equiv E_{gs}(S_{\min})-E_{gs}(S_{\min}+1)=\beta U[\sqrt{P(m,S_{\min})} - \sqrt{P(m,S_{\min}+1)}]. \tag{3}$$

As $P(m,1)\neq K(m,1)$, the question arises is whether Eq. (3) produces the odd-even effect seen in TBRE-*s* calculations. In fact Eq. (3) does produce the odd-even effect as $P(m,S)$ contains only $S(S+1)$ and $S^2(S+1)^2$ terms (note that these terms have odd-even effect). For example, for $\Omega=16$, the Δ values given by Eqs. (2) and (3), in units of βU, are 0.71, 1.95, 1.65, 2.66, 1.89, 2.9, 1.99, 3.02, 2.04, 3.08, 2.07, 3.12, 2.08, 3.13, 2.09 for $m=2-16$. Even when $U_0^2\neq U_1^2$, the odd-even effect is preserved by Eq. (3).

B. Ground state structure in TBRE-*s* with excess corrections

Alternatively, Kaplan *et al.* [10] considered shape corrections, in terms of the excess parameter γ_2, to the Gaussian form of $\rho^{m,S}(E)$. For a TBRE, as the third moment vanishes, the important shape parameter is the γ_2 defined by the fourth central moment $\mathcal{M}_4=\langle(H-\varepsilon)^4\rangle$; $\gamma_2=\mathcal{M}_4/\sigma^4-3$. Methods for deriving the expression for exact $\mathcal{M}_4$ for fixed (m,S) are available in literature [13,17] but they are unwieldy. However, the behavior of $\mathcal{M}_4$ and, hence, γ_2 can be understood by writing $V(2)$ in s_z representation and calculating averages, using the binary correlation method described in [1,18], over the (m_1,m_2) states introduced just above Eq. (1); here, instead of $U(N/2)\otimes SU(2)$, one is using the direct sum subgroup $U(N/2)\oplus U(N/2)$. In nuclear physics, with SU(2) giving isospin, this is referred as proton-neutron formulation [1]. The binary correlation results, with finite m and Ω corrections, for the ensemble averaged second and fourth moments are worked out for a general k-body Hamiltonian by Tomsovic [19]. Adopting his results to the present case, one has [with $m=m_1+m_2$ and $S_z=(m_1-m_2)/2$]

$$\langle[V(2)]^2\rangle^{m_1,m_2}=\binom{m_1}{2}\left[\binom{\Omega-m_1+2}{2}+1\right]U_1^2+\binom{m_2}{2}\left[\binom{\Omega-m_2+2}{2}+1\right]U_1^2+(m_1m_2) \times[(\Omega-m_1+1)(\Omega-m_2+1)+1][(U_0^2+U_1^2)/4],$$

$$\langle[V(2)]^4\rangle^{m_1,m_2}=2\{\langle[V(2)]^2\rangle^{m_1,m_2}\}^2+\sum_{k_1,k_2=0,1,2}\{f(m_1,\Omega,k_1,k_2)f(m_2,\Omega,2-k_1,2-k_2)+g(m_1,\Omega,k_1,k_2)g(m_2,\Omega,2-k_1,2-k_2)+g(m_1,\Omega,k_2,k_1)g(m_2,\Omega,2-k_2,2-k_1)+h(m_1,\Omega,k_1,k_2)h(m_2,\Omega,2-k_1,2-k_2)\}R_{k_1k_2};$$

$$f(m,\Omega,k_1,k_2)=\sum_{s=0}^{k_2}\binom{\Omega-m+k_1-s}{k_1}\binom{m-s}{k_1}\binom{m-s}{k_2-s}^2\binom{\Omega-m}{s}\binom{m}{s}\binom{\Omega+1}{s}\left(\frac{\Omega-2s+1}{\Omega-s+1}\right)\left[\binom{\Omega-s}{k_2}\binom{k_2}{s}\right]^{-1};$$

$$g(m,\Omega,k_1,k_2)=f(m,\Omega,k_1,k_2) \text{ without the } \binom{\Omega+1}{s} \text{ terms in the summation}$$

$$h(m,\Omega,k_1,k_2)=\sum_{s=0}^{<(k_1,k_2)}\binom{m-s}{k_1-s}\binom{m}{s}\binom{m-s}{k_2-s}\binom{\Omega-m}{s},$$

$$R_{k_1k_2}=[U_1^2]^{2-r}[(U_0^2+U_1^2)/4]^r, \quad r=(k_1 \mod 2)+(k_2 \mod 2);$$

$$\langle[V(2)]^p\rangle^{m,S}=[\langle\langle[V(2)]^p\rangle\rangle^{m,S_z=S}-\langle\langle[V(2)]^p\rangle\rangle^{m,S_z=S+1}]/d(m,S), \quad p=2,\ 4. \tag{4}$$

Let us point out that in the $\Omega\to\infty$ limit $\sigma^2(m,S)$ given by Eq. (4) agrees with Eq. (2) and also in the $\langle[V(2)]^4\rangle$ expression the f's with $s=k_2$ will dominate. Using Eq. (4), $\gamma_2(m,S)$ values are calculated in several examples. It is seen that γ_2 becomes more negative as S increases. For example, for $\Omega=20$ and $m=12$, the γ_2 values for $S=(0,1,2)$ are $(-0.386, -0.391, -0.401)$, $(-0.396, -0.401, -0.411)$, and $(-0.406, -0.413, -0.426)$ for U_0^2/U_1^2 taking values 0.3, 1, and 3, respectively. It should be pointed out that more negative is γ_2, the higher will be the ground state. Thus the trend seen in γ_2, though the variation is slow, do point out that the $S=0$ ground state will tend to be lower than $S=1$ ground state. Calculating the dimensions, variances, and γ_2 values using Eqs. (1), (2), and (4), the densities $d(m,S)\rho^{m,S}(E)$ are

constructed as corrected Gaussians using the so-called Cornish-Fisher method [20]. Now applying the Ratcliff procedure ground state energies and thereby Δ [see Eq. (3)] are calculated. It is seen that for $m=12$, 20 cases (with $\Omega=20$, $U_1^2/U_0^2=0.3$, 1, 3) the $\Delta\approx 0.01$ [in units of $\sigma^2(m,0)$] without γ_2 correction while it increases to ≈ 0.2 with the γ_2 correction. This result is consistent with the conclusions in Kaplan *et al*. work [10]. However, in this work the γ_2 variation is much larger (estimates of γ_2 for $S=0$, 1 for a four-particle system are given in [10] and they compare well with their numerical results) and this has to do with the fact that they consider paired states (for example, $a^\dagger_{i1/2}a^\dagger_{i-1/2}$ is a paired state for two particles and for m particles there can be a maximum of $m/2$ pairs in a given state) with fixed S. Group theory for calculating centroids, variances, and γ_2 for states with fixed m and S along with a given number of pairs will be considered elsewhere.

III. TWO-BODY RANDOM MATRIX ENSEMBLE IN A SINGLE *j* SHELL

The random matrix model TBRE-j is the other extreme to TBRE-s. Here one considers a single j shell with m-particles interacting via a J preserving two-body interaction (J is the total angular momentum of the m fermion system). With J a good quantum number, the Hamiltonian matrix divides into disconnected blocks with each of them labeled by J. The dimension $d(m,J)$ of the (m, J) block can be determined in many ways;

$$\sum_J (2J+1)d(m,J)=\binom{2j+1}{m}.$$

For a TBRE-j the two-particle matrix elements $V^{J_2}_{jjjj}=\langle (jj)J_2M_2|V(2)|(jj)J_2M_2\rangle$, $J_2=0,2,...,2j-1$ are random variables (note that J_2 is two-particle J value and the V's are independent of M_2). Without loss of generality the SP energy of the j orbit can be put to zero so that $H=V(2)$. It is easy to recognize that TBRE-j has the group symmetry $U(2j+1)\supset O(3)$ with all the m-particle states belonging to the antisymmetric representation of $U(2j+1)$ and $O(3)$ generating the J quantum number. Unlike in the TBRE-s case, here the scalar operators $\hat{n}$ and J^2 are not sufficient to write exact expressions for the moments of fixed-J densities $\rho^{m,J}(E)$. This is related to the so-called integrity basis operators for a group-subgroup chain $G\supset K$; see [21]. However, theory for good approximate formulas can be developed in the situation that the system is dilute [$j\to\infty$, $m\to\infty$, $m/(2j+1)\to 0$, and $J\ll(J)_{\max}$] and the Hamiltonian is complex [i.e., H belongs to TBRE]. Before discussing the results for fixed-J centroids and variances for TBRE-j, let us digress and address the more general problem of m fermions in several j orbits and the H is a J conserving $(1+2)$-body Hamiltonian.

A. Fixed-*J* centroids and variances from the bivariate $\rho(E,M)$ density

Let us consider m particles in many j orbits, then the number of SP states $N=\Sigma_j(2j+1)$. The centroid and variance of the state density $\rho_1(E)=\langle\delta(H-E)\rangle^m$ are $\varepsilon_H(m)=\langle H\rangle^m$ and $\sigma^2_H(m)=\langle[H-\varepsilon(m)]^2\rangle^m$, respectively. Similarly, for the M density $\rho_2(M)=\langle\delta(J_z-M)\rangle^m$ the centroid is zero and the variance $\sigma^2_{J_z}(m)=\langle J_z^2\rangle^m$. Let us define the standardized variables $\hat{E}=[E-\varepsilon_H(m)]/\sigma_H(m)$ and $\hat{M}=M/\sigma_{J_z}(m)$. Similarly, $\tilde{H}$ is the traceless part of H; $\langle\tilde{H}\rangle^m=0$. It is well established that for a TBRE [also for a $(1+2)$-body random matrix ensemble], not only $\rho_1(E)$ and $\rho_2(M)$ are close to Gaussian but also the joint bivariate density $\rho_{12}(E,M)=\langle\delta(H-E)\,\delta(J_z-M)\rangle^m$ is a bivariate Gaussian with lower order bivariate cumulant corrections [22]; note that here the bivariate correlation coefficient is zero. The corrected $\rho_{12}(E,M)$ is [with the third moment of $\rho_1(E)$ being zero for a TBRE]

$$\rho_{12}(E,M)=\Bigg\{1+\frac{k_{12}}{2}\hat{E}(\hat{M}^2-1)+\Bigg[\frac{k_{40}}{24}(\hat{E}^4-6\hat{E}^2+3)+\frac{k_{12}^2}{8}(\hat{E}^2-1)(\hat{M}^4-6\hat{M}^2+3)+\frac{k_{22}}{4}(\hat{E}^2-1)\times(\hat{M}^2-1)+\frac{k_{04}}{24}(\hat{M}^4-6\hat{M}^2+3)\Bigg]\Bigg\}\times\rho_{1;\mathcal{G}}(E)\rho_{2;\mathcal{G}}(M). \tag{5}$$

Equation (5) is central to our subsequent discussions. In Eq. (5) $\mathcal{G}$ stands for Gaussian and the bivariate cumulants k_{12} and k_{22} are given by

$$k_{12}(m)=\frac{\langle J_z^2\tilde{H}\rangle^m}{\sigma_H(m)\sigma^2_{J_z}(m)},\quad k_{22}(m)=\frac{\langle J_z^2\tilde{H}^2\rangle^m}{\sigma^2_H(m)\sigma^2_{J_z}(m)}-1. \tag{6}$$

Similarly $k_{40}(m)$ and $k_{04}(m)$ are the γ_2 values for $\rho_1(E)$ and $\rho_2(M)$ densities respectively. As pointed out in [22], fixed-M averages of powers of H can be written as integrals (with respect to E) involving $\rho_{12}(E,M)$. These and the identification [23] $\langle\langle H^p\rangle\rangle^{m,J}=-[(\partial/\partial M)\langle\langle H^p\rangle\rangle^{m,M}]_{M=J+1/2}$ lead to simple forms for fixed-J dimensions, centroids, and variances,

$$d(m,J)\simeq\binom{N}{m}\frac{(2J+1)}{\sqrt{8\pi}\sigma^3_{J_z}}\exp\left(-\frac{\left(J+\frac{1}{2}\right)^2}{2\sigma^2_{J_z}}\right),$$

$$\varepsilon(m,J)\simeq\left[\varepsilon_H(m)-\frac{3}{2}\sigma_H(m)k_{12}(m)\right]+\sigma_H(m)\frac{k_{12}(m)}{2\sigma^2_{J_z}(m)}J(J+1), \tag{7}$$

$$\sigma^2(m,J) \simeq \sigma_H^2(m)\left[1-\frac{3}{2}k_{22}(m)\right] + k_{22}(m)\frac{\sigma_H^2(m)}{2\sigma_{J_z}^2(m)}J(J+1).$$

A test of the results in Eq. (7) is carried out using the so-called $K+12\,fp$ interaction [24] in nuclear $(2s1d)^{12}$ space. In this example $\sigma_{J_z}^2(z)=12$, $\varepsilon_H(m)=-94.16$ MeV, $\sigma_H^2(m)=139\,\mathrm{MeV}^2$, $k_{12}(m)=-0.088$, and $k_{22}(m)=-0.12$. The exact (ε,σ) values, for example, for $J=0$, 4, 8, and 12 (in MeV) are $(-92.3, 13.3)$, $(-93.4, 12.3)$, $(-95.9, 10.2)$, and $(-100.5, 6.5)$ respectively. From Eq. (7) the corresponding results are $(-92.6, 12.9)$, $(-93.5, 12.2)$, $(-95.7, 10.4)$, and $(-99.3, 6.6)$, respectively. Thus the agreement between the approximate values given by Eq. (7) and the exact nuclear shell model results is excellent. Note that for the $J=0$, 4, 8, and 12 cases, the exact dimensionalities are 11 434, 49 441, 11 975, and 237. Now we will apply the results in Eq. (7) to TBRE-*j*, examine the structure of fixed-*J* centroids and variances and compare them with the results given by Mulhall *et al.* [4].

B. Structure of fixed-*J* centroids and variances in TBRE-*j*

Fixed-*J* centroids, as seen from Eq. (7), are basically determined for a TBRE by the k_{12} bivariate cumulant. Hence their structure in terms of the basic one-and two-particle matrix elements is determined by the structure of k_{12} in terms of these basic inputs. This problem is solved for TBRE-*j* by carrying out the U(N) decomposition of the Hamiltonian and that of the J^2 operator using Eq. (A1). Then the trace propagation Eq. (A2) give simple formulas for $\varepsilon_H(m)$, $\sigma_H^2(m)$, $\sigma_{J_z}^2(m)$, and $\langle J_z^2\tilde{H}\rangle^m$. For example the unitary decomposition of the J^2 operator and the expression for $\langle J_z^2\tilde{H}\rangle^m$ are,

$$J^2=(J^2)^{\nu=0}+(J^2)^{\nu=2},$$

$$(J^2)^{\nu=0}=\tfrac{1}{2}\hat{n}(2j+1-\hat{n})(j+1),$$

$$(J^2)^{\nu=2}\Leftrightarrow(J^2)^{\nu=2:J_2}_{jjjj}=J_2(J_2+1)-(2j-1)(j+1) \xrightarrow{j\gg 1} (-1)^{J_2+1}2j(j+1)(2j+1)\times\begin{Bmatrix} J_2 & j & j\\ 1 & j & j\end{Bmatrix},$$

$$\langle J_z^2\tilde{H}\rangle^m=\tfrac{1}{3}\langle J^2H^{\nu=2}\rangle^m=\frac{m(m-1)(N-m)(N-m-1)}{N(N-1)(N-2)(N-3)} \times\tfrac{1}{3}\sum_{J_2}(2J_2+1)V^{J_2}_{jjjj}[J_2(J_2+1) -(2j-1)(j+1)]; \quad N=(2j+1). \tag{8}$$

Note that for a single j shell there will be no $\nu=1$ parts for H and J^2 operators. It is useful to add that the propagator for $\langle J_z^2\tilde{H}\rangle^m$ in Eq. (8) reduces to m^2/N^2 in the dilute limit. Using Eqs. (6)–(8) and (A2) the final result for $\varepsilon(m,J)$ is easily obtained. In the dilute limit it takes the form,

$$\varepsilon(m,J)\rightarrow\left\{\left[\frac{m^2}{(2j+1)^2}\sum_{J_2}(2J_2+1)V^{J_2}_{jjjj}\right]-\left[\frac{3m}{2j(j+1)(2j+1)^2}\sum_{J_2}(2J_2+1)V^{J_2}_{jjjj}[J_2(J_2+1)-2j(j+1)]\right]\right\} + \frac{3}{2}\left\{\frac{\sum_{J_2}(2J_2+1)V^{J_2}_{jjjj}[J_2(J_2+1)-2j(j+1)]}{|j(j+1)(2j+1)|^2}\right\}J(J+1). \tag{9}$$

It is remarkable to see that, after neglecting the second piece of the first term (for large m the first piece dominates), the two terms in the formula (9) are nothing but the first two terms in the expression derived by Mulhall *et al.* see Eqs. (7)–(9) in [4]. Thus the cranking approximation and the use of Fermi occupancies employed in [4] is equivalent to the generation of the near bivariate Gaussian form in Eq. (5) for a TBRE. Therefore, one can use Eq. (5) to go beyond the simple TBRE-*j* to TBRE in several *j* shells (with H's preserving J).

Fixed (m, J) variances, as can be seen from Eq. (7), are determined by the k_{22} cumulant. With the unitary decomposition of the J^2 operator, it is seen that

$$k_{22}(m)=\langle(J^2)^{\nu=2}\langle H^{\nu=2}\rangle^2\rangle^m/\langle(J^2)^{\nu=0}\rangle^m\sigma_H^2(m).$$

Therefore, the only unknown quantity in determining $\sigma^2(m,J)$ is the m-particle average $\langle(J^2)^{\nu=2}(H^{\nu=2})^2\rangle^m$. This will be a sixth order polynomial in m as $(J^2)^{\nu=2}(H^{\nu=2})^2$ is a six-body operator. The particle-hole symmetry of the $\nu=2$ operators allows one to reduce the seven expansion coefficients of this polynomial into averages in two- and three-particle spaces. The final expression is obtained by simplifying the general results for $\langle U^{\nu=2}V^{\nu=2}W^{\nu=2}\rangle$ given in [1,25] and applying the approximation given in Eq. (8) for $(J^2)^{\nu=2}$. The result is,

$$\langle (J^2)^{\nu=2}(H^{\nu=2})^2\rangle^m = \left[\frac{m(m-1)(m-2)(m-3)(N-m)(N-m-1)}{N(N-1)(N-2)(N-3)(N-4)(N-5)} + \frac{m(m-1)(N-m)(N-m-1)(N-m-2)(N-m-3)}{N(N-1)(N-2)(N-3)(N-4)(N-5)}\right] \times (2j(j+1)(2j+1))A + \left[\frac{m(m-1)(m-2)(N-m)(N-m-1)(N-m-2)}{N(N-1)(N-2)(N-3)(N-4)(N-5)}\right](2j(j+1)(2j+1))B;$$

$$A = -\sum_{J_2} (2J_2+1)\begin{Bmatrix} J_2 & j & j \\ 1 & j & j \end{Bmatrix}(V^{\nu=2:J_2}_{jjjj})^2.$$

$$B = 8\sum_{J_1,J_2,J_3} (2J_1+1)(2J_2+1)(2J_3+1)\begin{Bmatrix} J_1 & j & j \\ 1 & j & j \end{Bmatrix} \times \begin{Bmatrix} j & j & J_3 \\ j & J_2 & j \\ J_1 & j & j \end{Bmatrix} V^{\nu=2:J_2}_{jjjj} V^{\nu=2:J_3}_{jjjj}.$$

$$N = (2j+1). \tag{10}$$

Equation (10) combined with Eqs. (7), (8), and (A2) will give $\sigma^2(m,J)$ in terms of $V^{J_2}_{jjjj}$. It is seen that the structure of $\sigma^2(m,J)$ is more complex [compared to $\varepsilon(m,J)$ in Eq. (9)] as Eq. (10) involves 6-*j* and 9-*j* symbols. Using the fixed (m,J) variances one can go beyond the centroids, i.e., beyond the investigations in [4], and study the role of the Gaussian widths in generating the dominance of $J=0$ ground states for a TBRE-*j*. Numerical calculations are carried out for TBRE-*j* with $j=\frac{17}{2}$ and $\frac{27}{2}$. They have confirmed that with centroids alone one gets somewhat more than 50% probability for ground states to appear with $J=0$. Equation (9) gives 50% probability but a correction to it, as pointed out in [4], enhances this probability. However, the moment variances are switched on [using Eqs. (7) and (10)], the effects of dimensions start dominating as we need to use the Ratcliff prescription for locating the lowest state for a given J value. Then it is seen that the $J=0$ dominance disappears and rarely $J=0$ states emerge as ground states. It is quite possible that: (i) one may have to include $[J(J+1)]^2$ terms in Eq. (7); (ii) just as in Sec. II B discussion, for TBRE-*j* also γ_2 effects need to be incorporated: (iii) as argued in [26] the $J=0$ states may be nongeneric and then constructing a corrected Gaussian form for locating the $J=0$ ground states may not be appropriate. Detailed investigations of (i)–(iii) in describing the $J=0$ dominance seen in numerical TBRE-*j* calculations, are beyond the scope of the present paper.

IV. CONCLUSIONS AND FUTURE OUTLOOK

The purpose of the results presented in this paper is to bring out the role of group symmetries in extended TBRE's. To this end investigations are carried out for the two ensembles TBRE-*s* and TBRE-*j*. Using the $U(N/2)\otimes SU(2)$ symmetry of TBRE-*s*, expressions for fixed (m, S) variances are easily derived. It is seen that the variances decrease as S increases. Relating the propagator $P(m,S)$ of the variances with the connectivity factor $K(m,S)$ it is shown that the Jacquod and Stone prescription for $\Delta \equiv E_{gs}(S_{min}) - E_{gs}(S_{min}+1)$ indeed gives the odd-even effect seen in numerical calculations. Going beyond the variances, using the binary correlation result for γ_2, it is shown that even with the Ratcliff prescription one sees the dominance of $S=0$ ground states for TBRE-*s*. In the case of TBRE-*j* it is shown that its $U(N)\supset O(3)$ group structure combined with the near bivariate Gaussian form of fixed-(E, M) densities lead to simple formulas for centroids and variances. Thus, for TBRE-*s* the group symmetry alone gives a simple structure for the centroids and variances (also γ_2—note that so far there is no binary correlation result for γ_2 of fixed-J densities) but in the case of TBRE-*j* in addition to the group symmetry one needs the random matrix nature of the ensemble as well as the dilute limit conditions. In particular it is shown that the expression for fixed-J centroids derived using the near bivariate Gaussian nature coincides with the result given in [4] where the cranking approximation and Fermi occupancies are used in the derivation. It is pointed out that the conclusions regarding the dominance of $J=0$ ground states in TBRE-*j* based on centroids alone may not remain valid once variances are also included and hence answering questions in TBRE-*j* (or TBRE with several *j*'s with H preserving J) requires much further study. It should be pointed out that the formulation in Sec. III can be applied to the more general TBRE-$(j_1 j_2 \ldots)$ (also called in literature TBRE-J; see [1]). Finally, it is expected that the investigations presented in this paper will lead to studies of TBRE's with a wider class of group symmetries and they, just as the TBRE-*s* [9], may find applications in Measoscopic Physics.

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APPENDIX

With respect to the $U(N)$ group a two-body interaction $V(2)$, defined by the two-particle matrix elements V_{ijkl}

$=_a\langle kl|V(2)|ij\rangle_a$, decomposes into scalar ($\nu=0$); effective one-body ($\nu=1$), and irreducible two-body ($\nu=2$) parts [27],

$$V^{\nu=0}=\frac{\hat{n}(\hat{n}-1)}{2}\overline{V};\quad \overline{V}=\binom{N}{2}^{-1}\sum_{i<j}V_{ijij},$$

$$V^{\nu=1}=\frac{\hat{n}-1}{N-2}\sum_{i,j}\zeta_{i,j}a_i^\dagger a_j;\quad \zeta_{i,j}=\left[\sum_k V_{kikj}\right]-\left[(N)^{-1}\sum_{r,s}V_{rsrs}\right]\delta_{i,j},$$

$$V^{\nu=2}=V-V^{\nu=0}-V^{\nu=1}\Leftrightarrow V_{ijkl}^{\nu=2}. \tag{A1}$$

Similarly a one-body Hamiltonian $h(1)=\Sigma_i\hat{n}_i\epsilon_i$ decomposes into $h^{\nu=0}=\hat{n}\overline{\epsilon}$ and $h^{\nu=1}=\Sigma_i\hat{n}_i\epsilon_i^1$ where $\overline{\epsilon}=N^{-1}\Sigma_i\epsilon_i$ and $\epsilon_i^1=\epsilon_i-\overline{\epsilon}$. Now the m-particle variances for $H=h(1)+V(2)$ are given by

$$\sigma^2(m)=\frac{m(N-m)}{N(N-1)}\sum_{ij}\left\{\epsilon_i^1\delta_{i,j}+\frac{m-1}{N-2}\zeta_{i,j}\right\}^2+\frac{m(m-1)(N-m)(N-m-1)}{N(N-1)(N-2)(N-3)}\langle\langle(V^{\nu=2})^2\rangle\rangle^{m=2} \tag{A2}$$

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